ASTRON 449, Winter 2019 – Problem Set 2

Due Thu Jan. 31, in class.

REGULAR PROBLEMS:

1. Filling the gaps in class derivations. a) Prove Newton's first and second theorems for spherical systems using Gauss' theorem.

b) Starting from the expression derived in class,

$$W = -\frac{1}{8\pi G}\int d^3\mathbf{x}|\nabla\Phi|^2, \tag{1}$$

show that the gravitational potential energy can also be written as

$$W = \frac{1}{2}\int d^3\mathbf{x}\rho(\mathbf{x})\Phi(\mathbf{x}). \tag{2}$$

2. BT2, problem 2.3. Show that the potential of an infinite razor-thin sheet of surface density Σ in the plane $z = 0$ is $\Phi = 2\pi G\Sigma|z| + \text{const.}$,

a) using Gauss's theorem, and

b) from Poisson's equation.

3. NFW vs. Hernquist profiles. Consider a dark matter halo well approximated by a viral mass $M_{200\mathrm{c}}$ and NFW concentration parameter $c = R_{200\mathrm{c}}/r_\mathrm{s}$ (where r_s is the halo scale radius). By definition, $M_{200\mathrm{c}} = 4\pi\int_0^{R_{200c}} drr^2\rho(r)$ is the mass contained within a radius R_{200c} whose mean enclosed density is 200 times the critical density of the Universe, ρ_crit.

a) Suppose first that the halo density profile $\rho(r)$ exactly follows an NFW profile. Show that the total mass integrated to $r = \infty$ diverges, i.e. $M(\infty) \equiv 4\pi\int_0^\infty drr^2\rho(r) \to \infty$.

b) Suppose now that the density profile is approximated by an Hernquist model. Show that the total mass of the Hernquist model (integrated to infinity) is always finite.

c) To avoid the divergent total mass implied by the NFW profile, dark matter halos are sometimes approximated with Hernquist profiles of the form

$$\rho(r) = \frac{M}{2\pi}\frac{a}{r}\frac{1}{(r+a)^3}, \tag{3}$$

even though the NFW profile is a more accurate fit to the halos forming in cold dark matter cosmological simulations. To do this, it is necessary to determine the Hernquist scale radius a that corresponds to an NFW halo of a given concentration. Identifying the total mass of the Hernquist profile with M_{200c} (i.e., setting $M = M_{200\mathrm{c}}$), derive an expression for a such that in the inner parts

where galaxies are located (small r), the Hernquist model matches the NFW halo being approximated. Express your answer in terms of r_s and c only.

4. Inner mass profiles of dwarf galaxies. In certain low-mass dwarf galaxies, the observed rotation curve rises close to linearly at small radii ($v_c \propto r$ for small r). Assume that dark matter dominates the potential in these regions (a reasonable assumption given the low efficiency of star formation in dwarf halos).

a) What does this imply for the total mass density versus radius, $\rho(r)$? You can assume spherical symmetry.

b) Compare your result with the inner ($r \to 0$) density profile expected in dark matter halos that follow a NFW profile. Are observations of dwarf galaxies consistent with NFW in the inner parts?

c) If there is a discrepancy, speculate on physical mechanisms for how these dwarf galaxy observations may be reconciled with the predictions of cold dark matter cosmologies. Since this is an active area of research beyond, you do not need to prove your answer and you will get credit for effort.

COMPUTATIONAL PROBLEMS:

Reminder concerning units: Treat Newton's constant G as a variable whose value can be modified in the code. By default, we work in dimensionless units and set $G = 1$.

The following computational problems build on the code that you developed for the first problem set (PS1). You will extend your orbit integration code to allow you to explore different numerical integration methods and more complex analytic potentials.

C1. Leapfrog and Runge-Kutta integrators. a) Implement the option `integr=leapfrog` in the Python program `integrate_analytic.py` that you wrote in PS1, i.e. such that a command of the form

```
python integrate_analytic.py x0 y0 z0 vx0 vy0 vz0 pot leapfrog t dt output_file
```

will use the leapfrog integrator with time-step dt to integrate an orbit in the specified potential.

b) Implement the option `integr=rk4`, which will do the same but for the 4th-order Runge-Kutta integrator.

c) Consider the same initial conditions for a circular orbit in a Keplerian potential that you used for the Euler integrator for PS1:

$$M = 1 \tag{4}$$
$$(\texttt{x0},\ \texttt{y0},\ \texttt{z0},\ \texttt{vx0},\ \texttt{vy0},\ \texttt{vz0}) = (1.0,\ 0.0,\ 0.0,\ 0.0,\ 1.0,\ 0.0).$$

Produce a plot similar to the one you produce in PS1 (three panels: orbit in the xy plane, fractional energy error, fractional angular momentum error), but evolving the test particle for 100 orbital periods in the Keplerian potential with time-steps $dt = 1,\ 0.1, 0.01,\ 0.001$ with the leapfrog integrator rather than the Euler method.

Note that, relative to PS1, we have added the long time-step $dt = 1$ (a substantial fraction $1/2\pi$ of a full orbital period), which would produce very poor results for the Euler integrator. You will likely have to use logarithmic vertical axes to show the full ranges of fractional errors for this problem.

d) Produce another plot for the same test (same choices of time-step) but for the 4th-order Runge-Kutta integrator instead of leapfrog.

e) Discuss how the energy and angular momentum errors decrease with decreasing dt with the leapfrog and Runge-Kutta integrators. Comment on how the differences can be understood in terms of properties of these integrators that we discussed in class.

C2. Orbits in a triaxial potential. Consider the following triaxial generalization of a Plummer model:

$$\phi(\mathbf{x}) = -\frac{GM}{\sqrt{x^2 + (q_1 y)^2 + (q_2 z)^2 + b^2}}, \tag{5}$$

where q_1 and q_2 are axis ratio constants. Set $M = 1$, $q_1 = 0.9$, $q_2 = 0.5$, and $b = 0.5$. This potential is neither spherically nor axially symmetric.

a) Do you expect the energy of a test particle to be conserved in this potential? What about the angular momentum? If not, why not?

b) Implement the option `pot=triplum` to enable the integration of orbits in this triaxial potential (the values of M, q_1, q_2, and b can be set in your .py source code).

c) Consider the initial conditions

$$(\texttt{x0},\ \texttt{y0},\ \texttt{z0},\ \texttt{vx0},\ \texttt{vy0},\ \texttt{vz0}) = (1.0,\ 0.0,\ 0.0,\ 0.0,\ 1.0,\ 0.1). \tag{6}$$

Focus now on the leapfrog integrator, which is commonly used in practical applications, and integrate the orbit for the same total time ($t = 200\pi$) as in problem C1 above. Do this for time-steps $dt = 0.1,\ 0.01,\ 0.001$.

Plot the projections of the numerical orbits in the xy and xz planes, as well as the fractional changes in energy and magnitude of the angular momentum versus time. Evaluate these changes relative to the initial values of the energy and angular momentum. As before, evaluate the angular momentum with respect to the origin and plot the curves for the different dt on the same panels for direct comparison.

d) Note the complex the orbital trajectories, which are representative of orbits in triaxial potentials. Examine how the orbital path, energy, and angular momentum change with decreasing dt, and use

this information to assess whether the orbits that you have computed are well converged, i.e. whether the apparent orbital complexity is a property of the "true" or "exact" orbit, or is affected by inaccuracies in the numerical integrations.